An investigation into Invariance (Part 2)

In this activity you will explore how repeated eigenvalues affect the geometry of transformations. Recall that eigenvectors have their direction preserved under transformation, and they scale by their associated eigenvalue.

- Open Vectorama (www.korovatron.co.uk/vectorama)
- Select **2D** and **ARROW** mode

Exercise 1

- Add a **2D** matrix that enlarges scale factor 2.
- Add a **UNIT SQUARE** from the **VECTOR PRESETS**.
- Apply the matrix.
- Open the matrix information panel [i]
- Inspect the **eigenvalues** and **eigenvectors** and toggle the **eigenspace** on.
- Add several vectors anywhere, in any direction. Apply the matrix.

Which vectors are **eigenvectors**? Can you explain your answer?
In this matrix, the repeated eigenvalue gives the same scale factor 2 in every direction. Since every vector is multiplied by the same scalar, direction is preserved for all vectors, so every non-zero vector is an eigenvector.

Exercise 2

- Switch to **3D.**
- Add a **3D** matrix that enlarges scale factor 3.
- Add a **UNIT CUBE** from the **VECTOR PRESETS**.
- Apply the matrix.
- Open the matrix information panel [i]
- Inspect the **eigenvalues** and **eigenvectors** and toggle the **eigenspace** on.
- Add several vectors anywhere, in any direction. Apply the matrix.

Which vectors are **eigenvectors**? Can you explain your answer?
In this matrix, the repeated eigenvalue gives the same scale factor 3 in every direction. Since every vector is multiplied by the same scalar, direction is preserved for all vectors, so every non-zero vector is an eigenvector.

Exercise 3

- Add the matrix $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$
  Inspect the **eigenvalues** and **eigenvectors**. Notice there are only two distinct **eigenvalues**, but three distinct **eigenvectors**.
- Add some multiples of the **eigenvector** for the unique **eigenvalue.**
- Add several vectors of the form $\alpha \boldsymbol{v}_2 + \beta \boldsymbol{v}_3$, where α, β are scalars and $\boldsymbol{v}_2, \boldsymbol{v}_3$ are the two **eigenvectors** with the same **eigenvalue**.
- Toggle the **eigenspace** on and apply the matrix. (You may need to zoom out and rotate for the best view).

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Describe what you have observed.

There are two distinct eigenvalues but 3 independent eigenvectors. Therefore, the two eigenvectors with the same eigenvalue scale vectors by the same factor, preserving direction. Therefore, the span of the two eigenvectors with the same eigenvalue forms a plane of eigenvectors. There is also an eigenline from the third eigenvector.

Exercise 4

- Add the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$
- Inspect the **eigenvalues** and **eigenvectors** and **eigenspace**.

How is this example different to Exercise 3?

There are only two independent eigenvectors, therefore two eigenlines. Repeated eigenvalue here does not create an eigenplane because there is only one independent eigenvector direction for that eigenvalue.

Exercise 5

- Add the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$
- Inspect the **eigenvalues** and **eigenvectors** and **eigenspace**.

Compare this with Exercise 2. What do you notice?

Also has a single distinct eigenvalue, but this time only one independent eigenvector, so the eigenspace is a single eigenline.

Exercise 6

Complete the table below

Exercise	2D/3D	Distinct eigenvalues	Independent eigenvector Directions	Shape of eigenspace(s)
1	2D	1	2	All Space R^2
2	3D	1	3	All Space R^3
3	3D	2	3	eigenplane & eigenline
4	3D	2	2	Two eigenlines
5	3D	1	1	Single eigenline

What conclusions can you make regarding the number of **distinct eigenvalues** and the number of **independent eigenvectors** in relation to **eigenspace**, for any given matrix transformation?

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Independent eigenvectors with **distinct** eigenvalues each produce an eigenline. This is because the distinct eigenvalues cause each line to have a different scaling factor, therefore only preserving direction along that line.

In 2D, if two independent eigenvectors have the same eigenvalue, then we have an eigenplane, spanning those two vectors, which cover all R^2.

In 3D, if three independent eigenvectors have the same eigenvalue, then the eigenspace is R^3.

In 3D, if two independent eigenvectors have the same eigenvalue then the plane spanning those two vectors forms an eigenplane. The third eigenvector forms an eigenline.